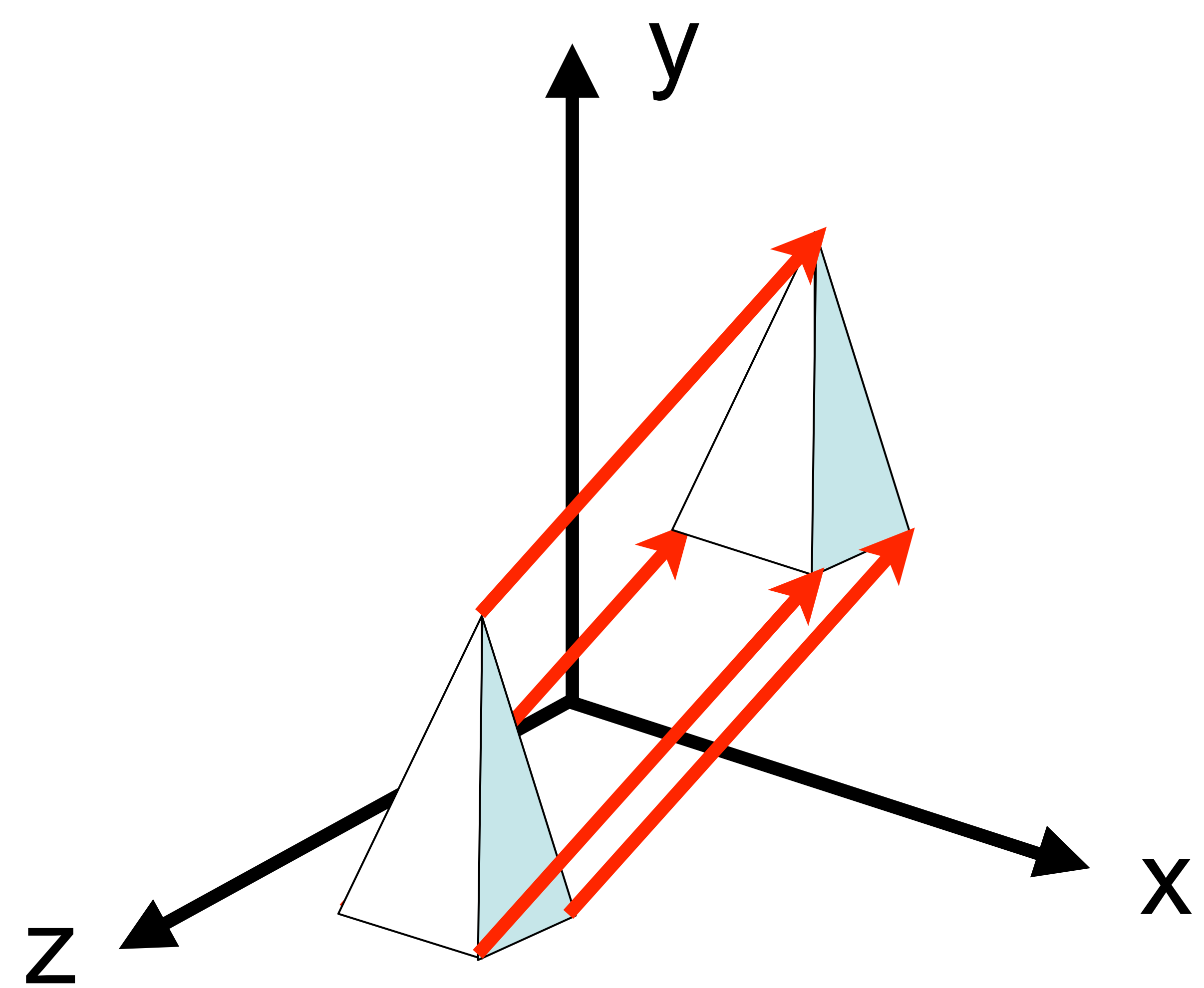
Chapter 2 - Basic Mathematics for 3D Computer Graphics

- Three-Dimensional Geometric Transformations
- Affine Transformations and Homogeneous Coordinates
- Combining Transformations

Translation

- Add a vector *t*
- Geometrical meaning: Shifting
- Inverse operation?
- Neutral operation?

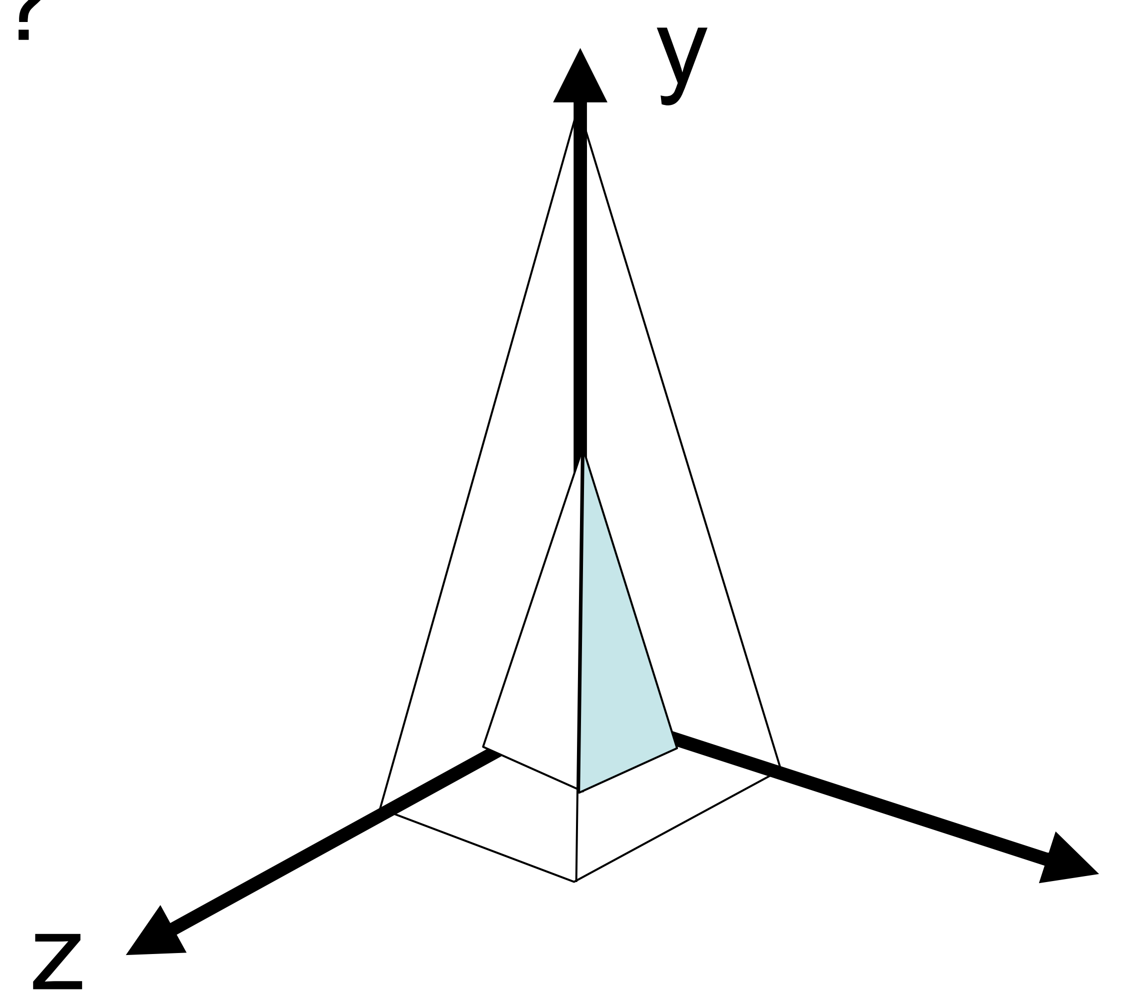
$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} p_1 + t_1 \\ p_2 + t_2 \\ p_3 + t_3 \end{pmatrix}$$



Uniform Scaling

- Multiply with a scalar *s*
- Geometrical meaning: Changing the size of an object
- What happens when we scale objects which are not at the origin?
- How can we fix that?

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \cdot s = \begin{pmatrix} p_1 \cdot s \\ p_2 \cdot s \\ p_3 \cdot s \end{pmatrix}$$

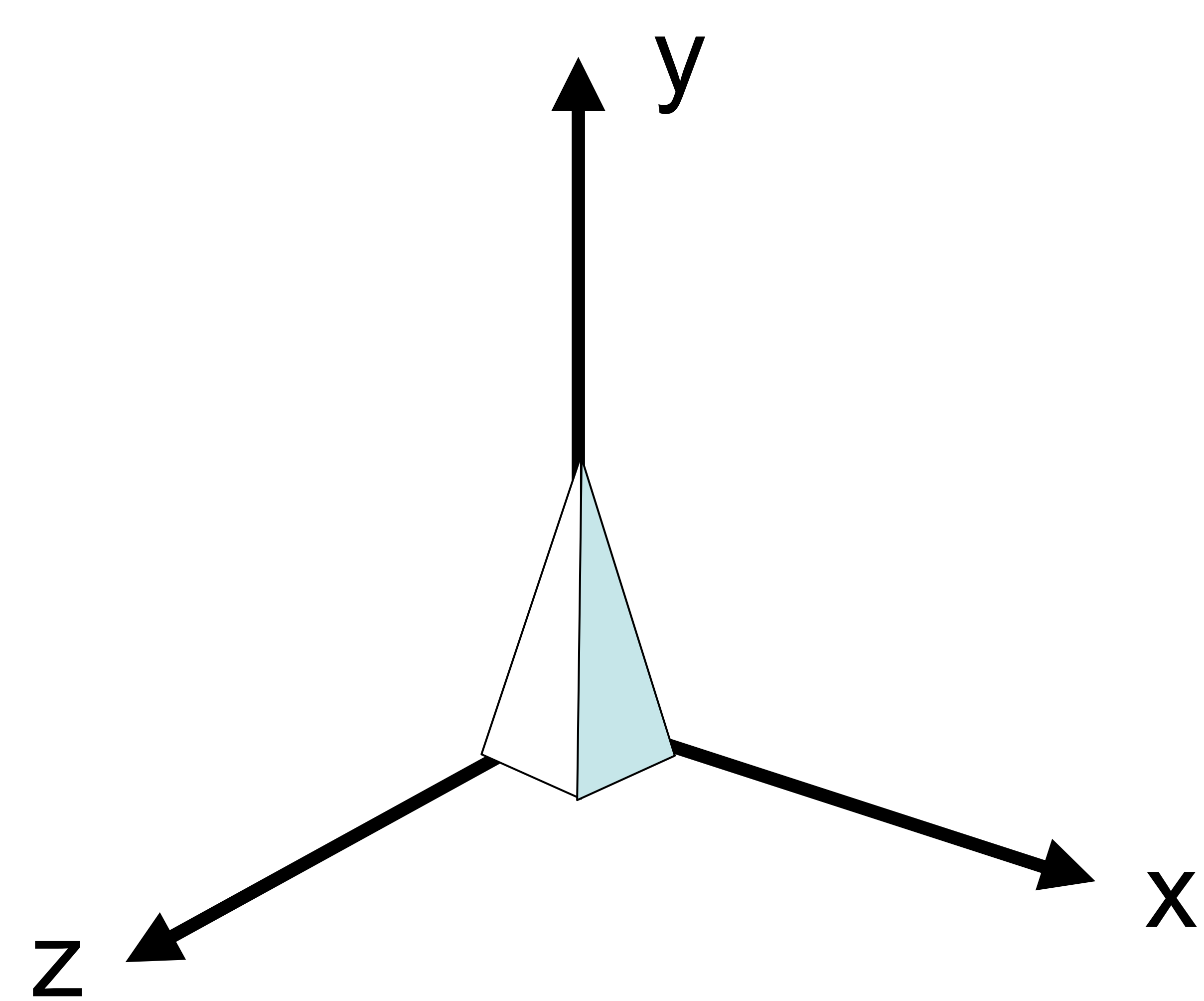


Non-Uniform Scaling

$$\begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 \cdot s_1 \\ p_2 \cdot s_2 \\ p_3 \cdot s_3 \end{pmatrix}$$

- Multiply with three scalars
- One for each dimension
- Geometrical meaning?

http://en.wikipedia.org/wiki/Utah_teapot



Reflection (Mirroring)

$$\begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 \cdot s_1 \\ p_2 \cdot s_2 \\ p_3 \cdot s_3 \end{pmatrix}$$

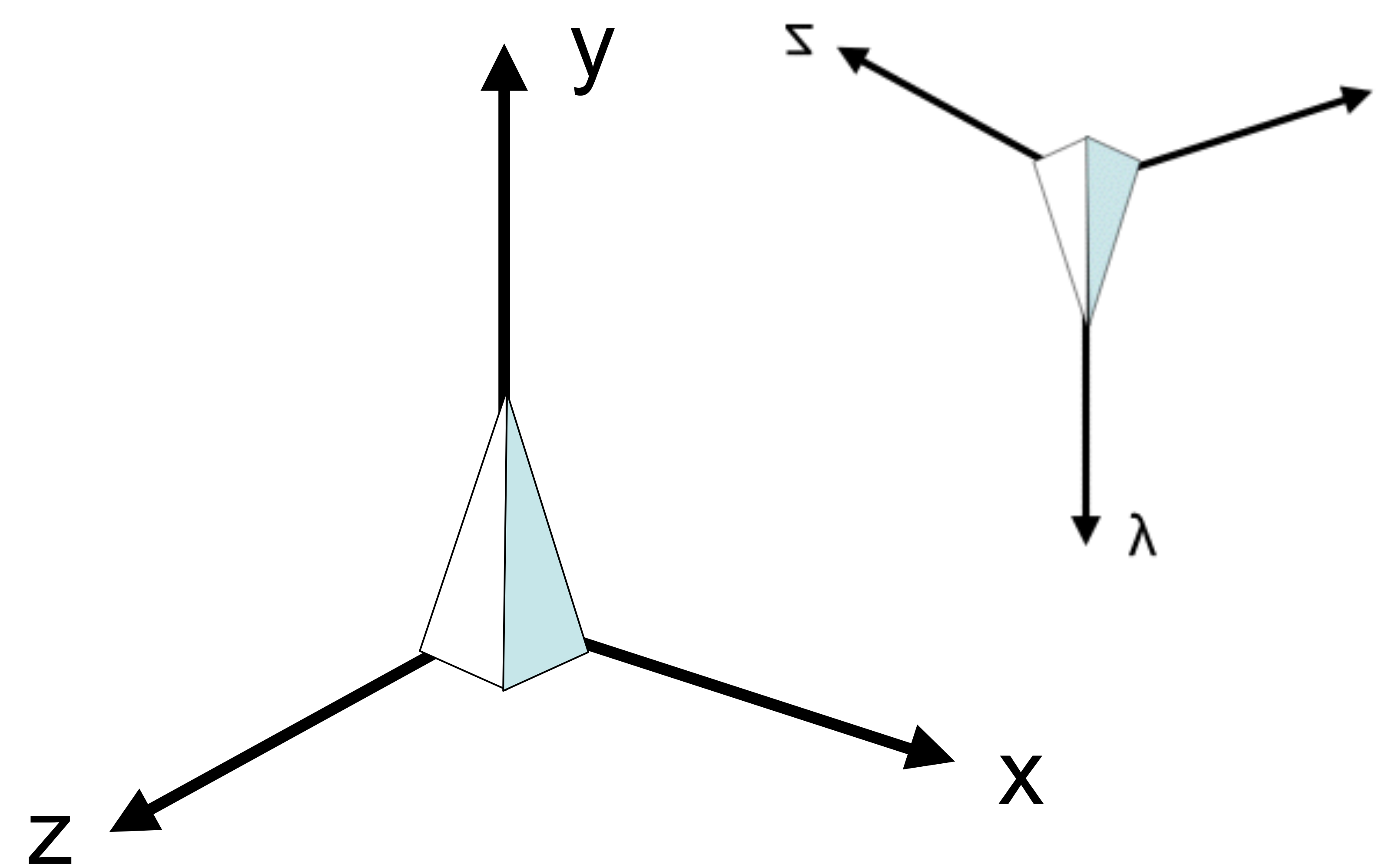
- Special case of scaling

$$s_1 \cdot s_2 \cdot s_3 < 0$$

- Example:

$$s_1 = 1,\ s_2 = -1,\ s_3 = 1$$

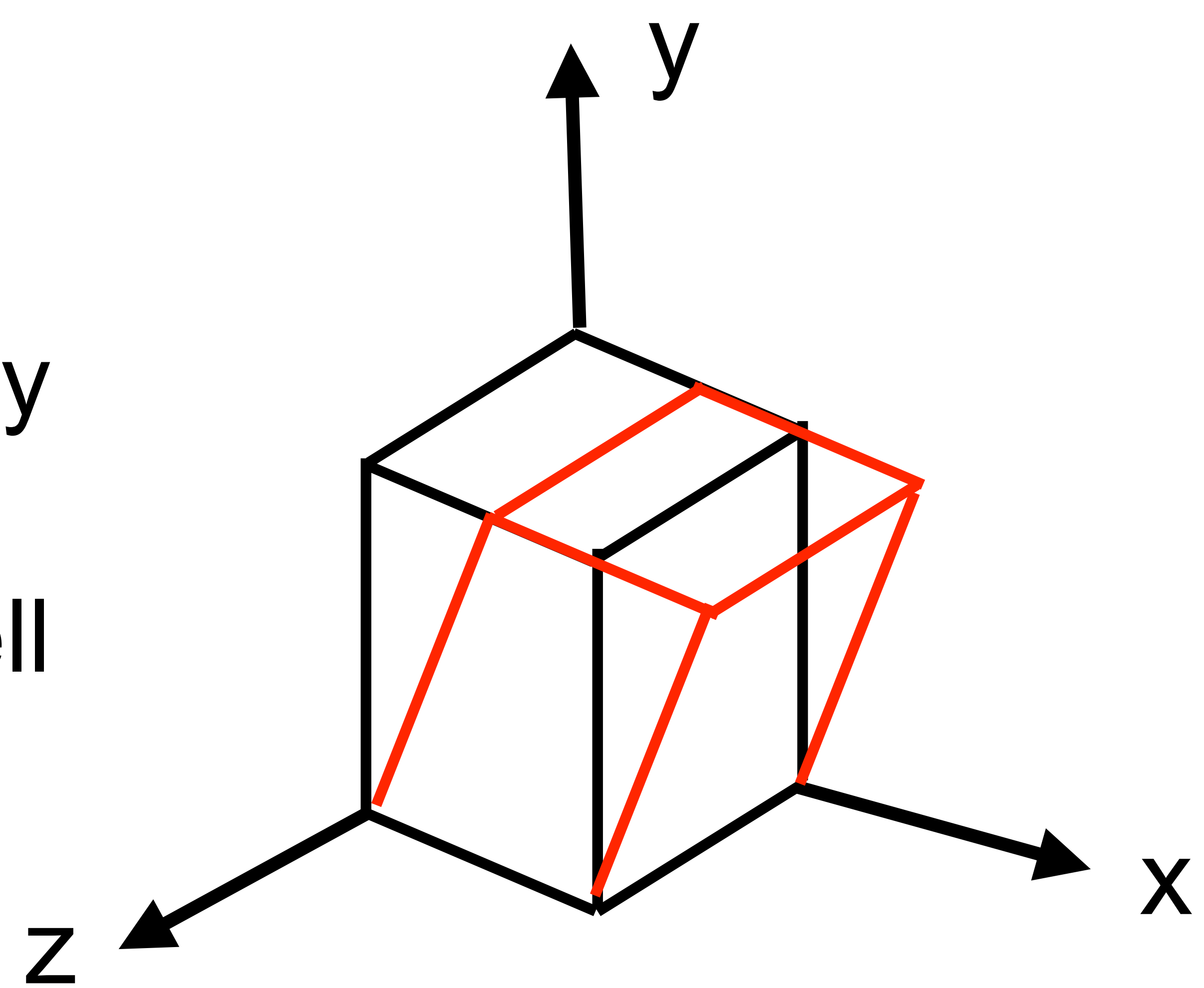
- Discuss: What does this mean for
  - surface normals?
  - order of polygon edges?
  - handedness?



Shearing along X Axis

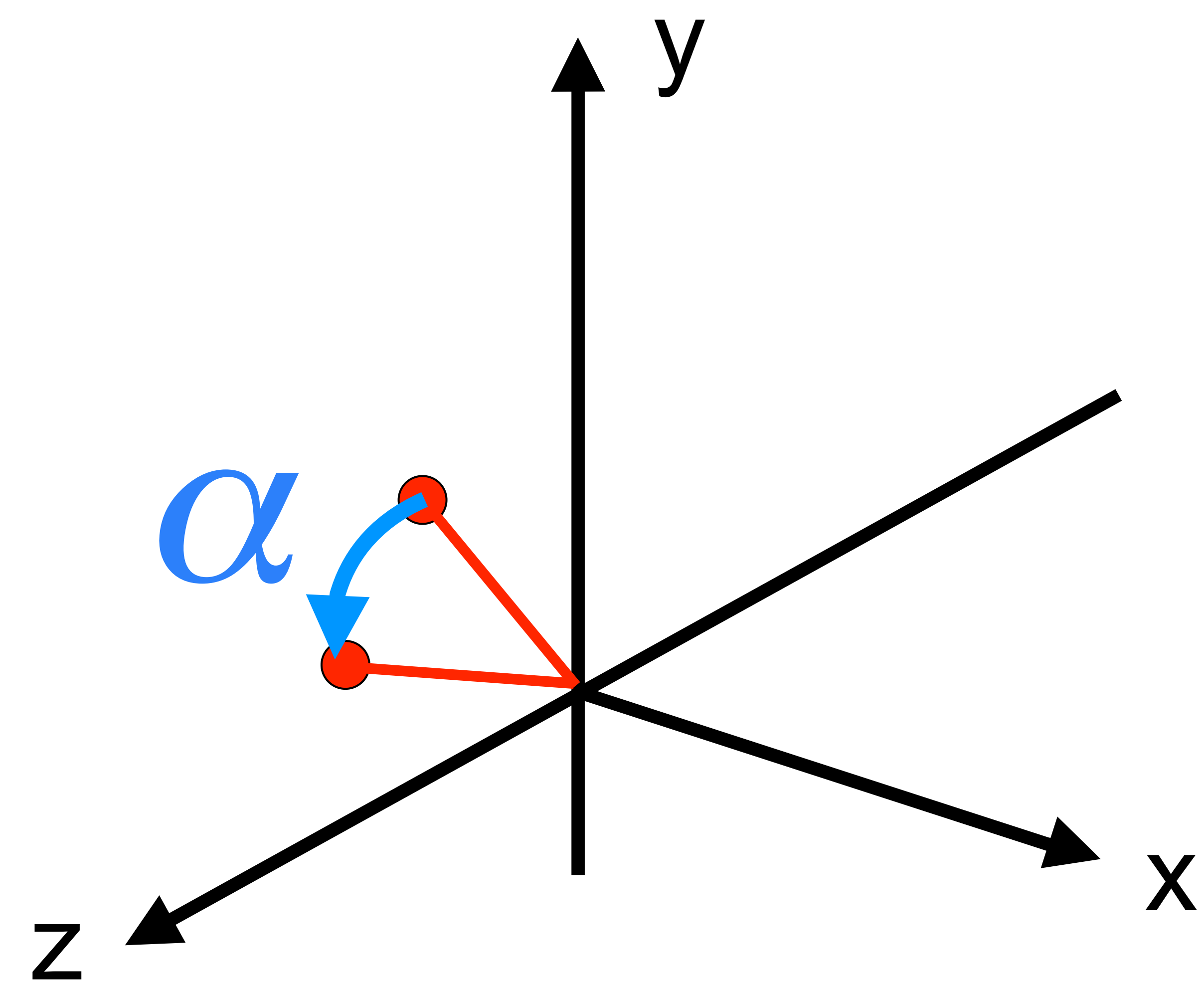
$$\begin{pmatrix} p_1 + m \cdot p_2 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

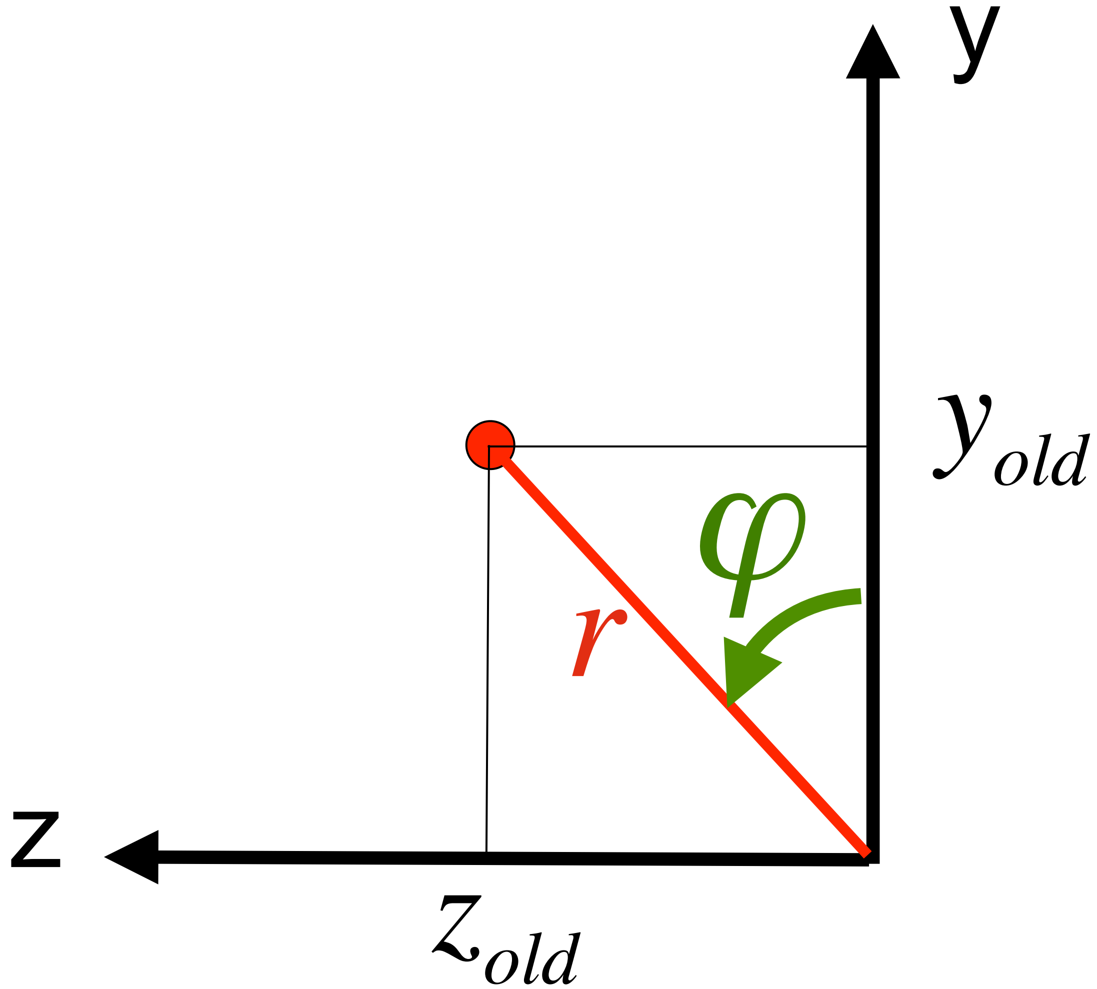
- Only x coordinate values are modified
- Modification depends linearly on y coordinate value
- Areas in x/y and x/z plane, as well as volume remain the same
- Generalization to other axes and arbitrary axis: see later...



Rotation about X Axis (1)

- x coordinate value remains constant
- Rotation takes place in y/z-plane (2D)
- How to compute new x and z coordinates from old ones?

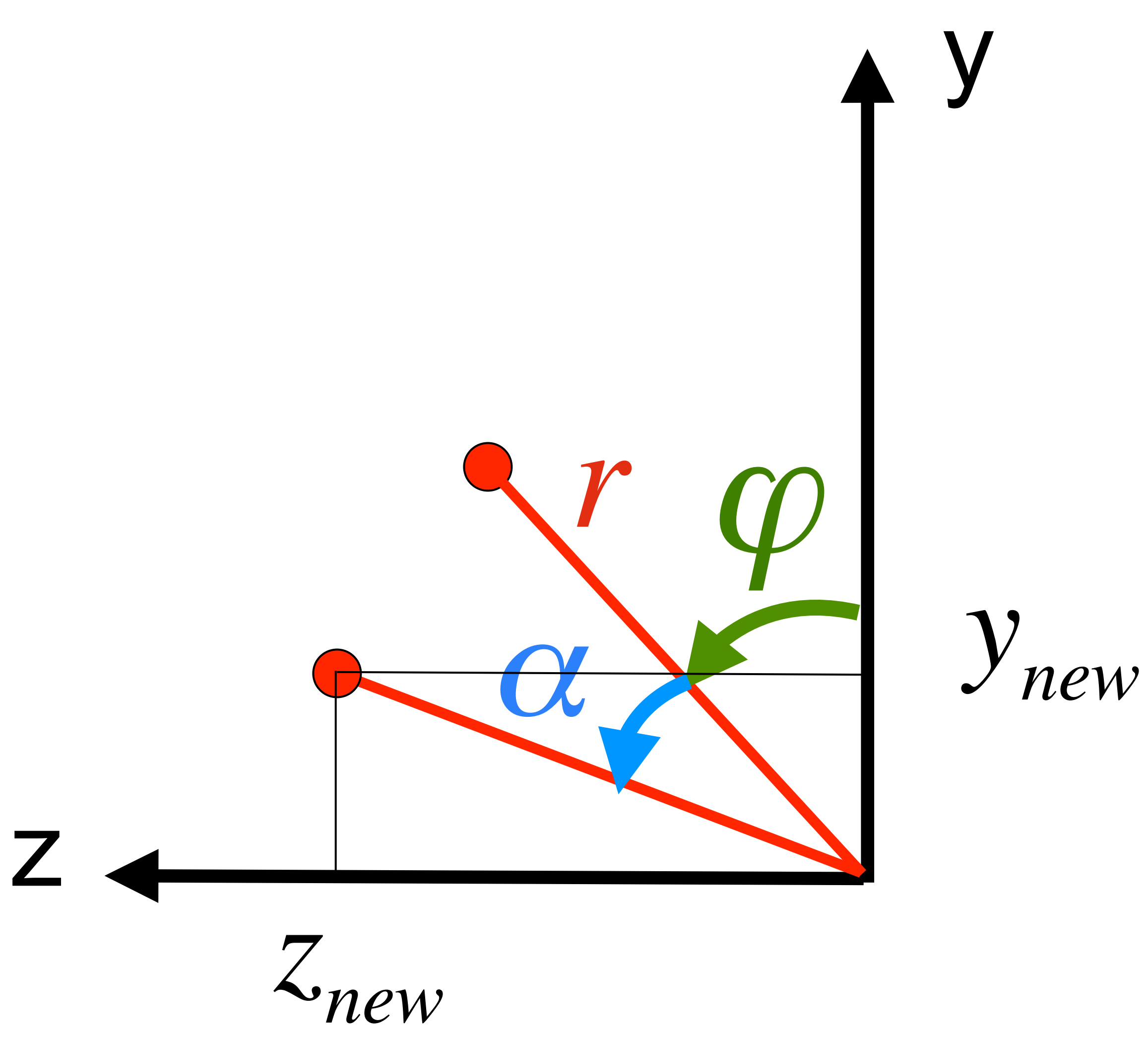




$$\sin\varphi = \frac{z_{old}}{r} \qquad z_{old} = r \cdot \sin\varphi$$

$$\cos\varphi = \frac{y_{old}}{r} \qquad y_{old} = r \cdot \cos\varphi$$

Rotation about X Axis (2)



$$y_{old} = r \cdot \cos\varphi$$

$$z_{old} = r \cdot \sin\varphi$$

$$\cos(\alpha + \varphi) = \frac{y_{new}}{r}$$

$$\begin{aligned} y_{new} &= r \cdot \cos(\alpha + \varphi) \\ &= r \cdot \cos a \cdot \cos\varphi - r \cdot \sin\alpha \cdot \sin\varphi \\ &= \cos\alpha \cdot y_{old} - \sin\alpha \cdot z_{old} \end{aligned}$$

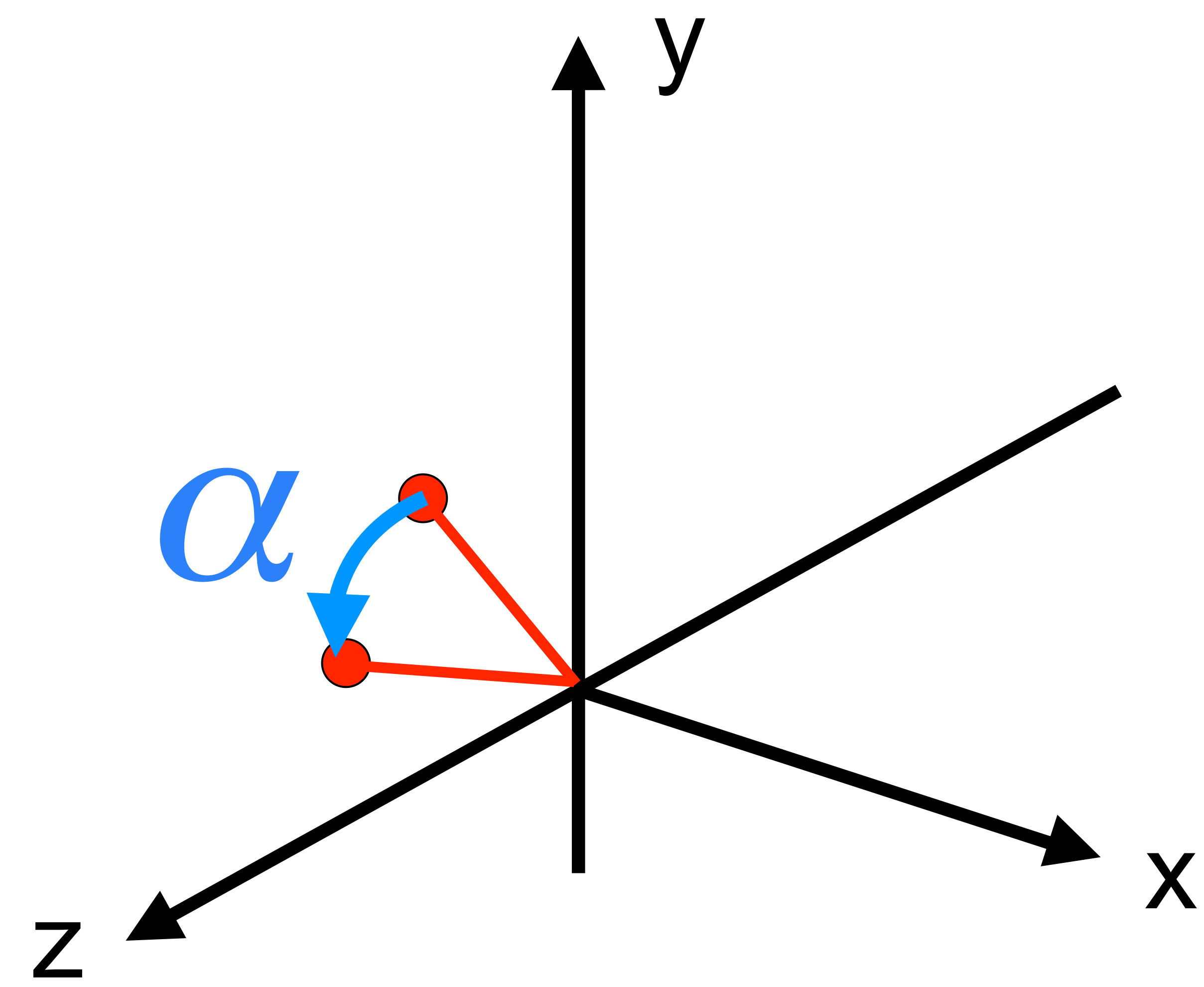
$$\sin(\alpha + \varphi) = \frac{z_{new}}{r}$$

$$\begin{aligned} z_{new} &= r \cdot \sin(\alpha + \varphi) \\ &= r \cdot \sin a \cdot \cos\varphi + r \cdot \cos\alpha \cdot \sin\varphi \\ &= \sin\alpha \cdot y_{old} + \cos\alpha \cdot z_{old} \end{aligned}$$

Rotation about X Axis (3)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ \cos\alpha \cdot p_2 - \sin\alpha \cdot p_3 \\ \sin\alpha \cdot p_2 + \cos\alpha \cdot p_3 \end{pmatrix}$$

- Special cases,
  e.g. 90 degrees, 180 degrees?
- How to rotate about other axes?



Elementary rotations

- Combine to express arbitrary rotation
- This is not always intuitive
- Order matters (a lot!)
- Likely source of bugs!

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ \cos\alpha \cdot p_2 - \sin\alpha \cdot p_3 \\ \sin\alpha \cdot p_2 + \cos\alpha \cdot p_3 \end{pmatrix}$$

$$\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \cos\beta \cdot p_1 + \sin\beta \cdot p_3 \\ p_2 \\ \cos\beta \cdot p_3 - \sin\beta \cdot p_1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \cos\chi \cdot p_1 - \sin\gamma \cdot p_2 \\ \sin\gamma \cdot p_1 + \cos\gamma \cdot p_2 \\ p_3 \end{pmatrix}$$

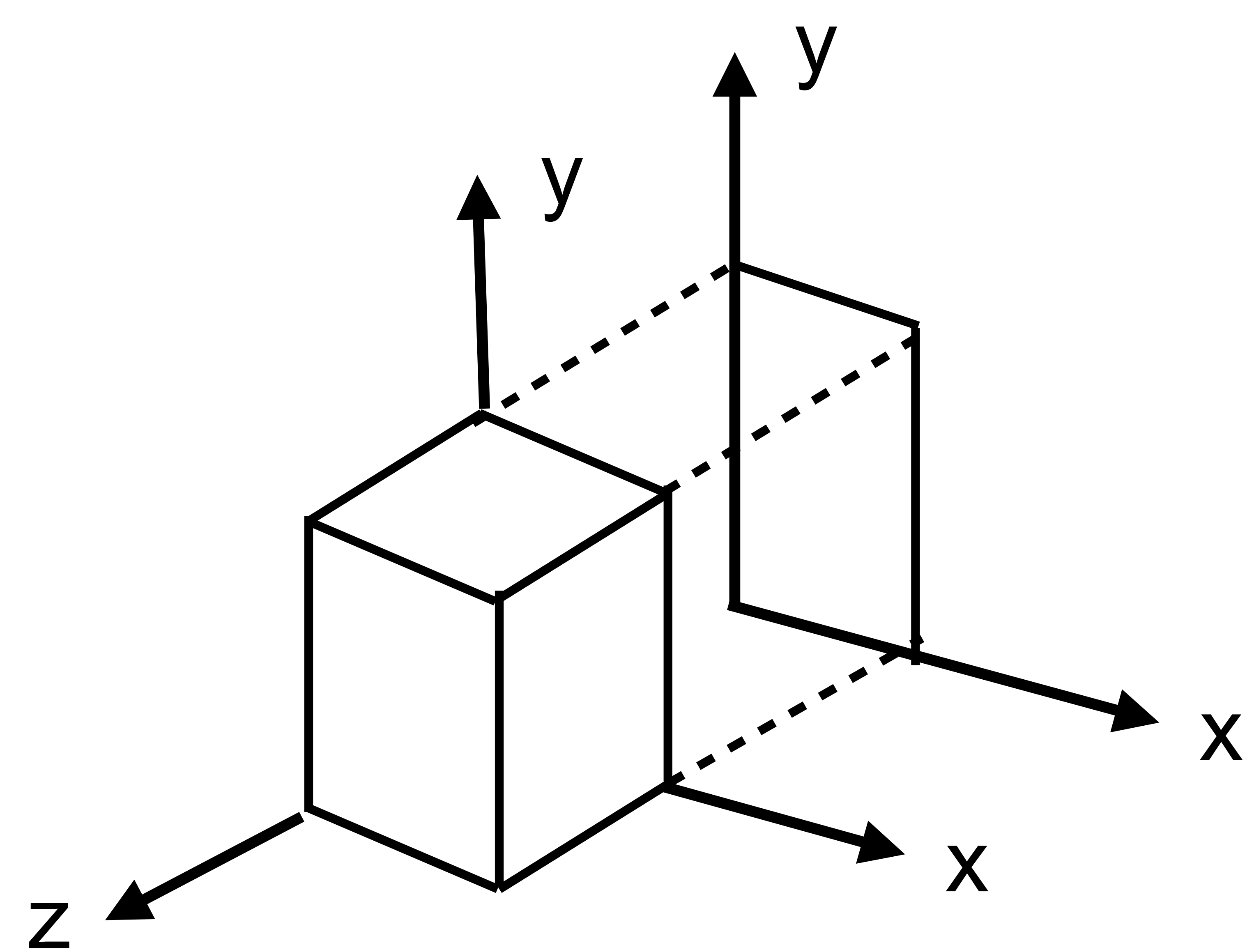
Transformation of Coordinate Systems

- Applying a geometric transformation...
  - ...to all points of a single object: Transforming the object within its own coordinate system.
  - ...to all points of all objects of the “world”: effectively transforming the reference coordinate system in the opposite direction!
- Geometric transformations can be used to…
  - …modify an object
  - ...place an object within a reference coordinate system
  - ...switch to different reference coordinates

Transformation from 3D to 2D: Projection

- Many different projections exist (see later)
- Projection onto x/y plane:
  - "Forget" the z coordinate value
- Other projections?
- Other viewpoints?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$



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Affine Transformation

- Mathematically: A transformation preserving collinearity
  - Points lying on a line before are on a line after transformation
  - Ratios of distances are preserved (e.g. midpoint of a line segment)
  - Parallel lines remain parallel
  - Angles and lengths are *not* preserved!
- Basic transformations: translation, rotation, scaling and shearing
  - All combinations of these are affine transformations again
  - Combination is associative, but not commutative
- General form of computation:
  - New coordinate values are defined by linear function of the old values

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = Ap + t$$

Combining Multiple Transformations

- Rotation, scaling and shearing are expressed as matrices
  - Associative, hence can all be combined into one matrix
  - **Many** of these operations can also be combined into one matrix

- Translation is expressed by adding a vector
  - Adding vectors is also associative
  - Many translations can be combined into a single vector

- Combination of Translation with other operations?
  - Series of matrix multiplications and vector additions, difficult to combine
  - How about using a matrix multiplication to express translation ?!?
  -
  -
  -

Homogeneous Coordinates

- Usage of a representation of coordinate-positions with an extra dimension
  - Extra value is a *scaling factor*
- 3D position (x, y, z) is represented by (x_h, y_h, z_h, h) such that

$$x = \frac{x_h}{h}, \quad y = \frac{y_h}{h}, \quad z = \frac{z_h}{h}$$

- Simple choice for scaling factor h is the value 1
  - In special cases other values can be used
- 3D position (x, y, z) is represented by $(x, y, z, 1)$

Translation Expressed in Homogeneous Coordinates

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} p_1 + t_1 \\ p_2 + t_2 \\ p_3 + t_3 \end{pmatrix}$$

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix} = \begin{pmatrix} p_1 + t_1 \\ p_2 + t_2 \\ p_3 + t_3 \\ 1 \end{pmatrix}$$

Scaling Expressed in Homogeneous Coordinates

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} s_1 p_1 \\ s_2 p_2 \\ s_3 p_3 \end{pmatrix}$$

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \\ 1 \end{pmatrix} = \begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix} = \begin{pmatrix} s_1 p_1 \\ s_2 p_2 \\ s_3 p_3 \\ 1 \end{pmatrix}$$

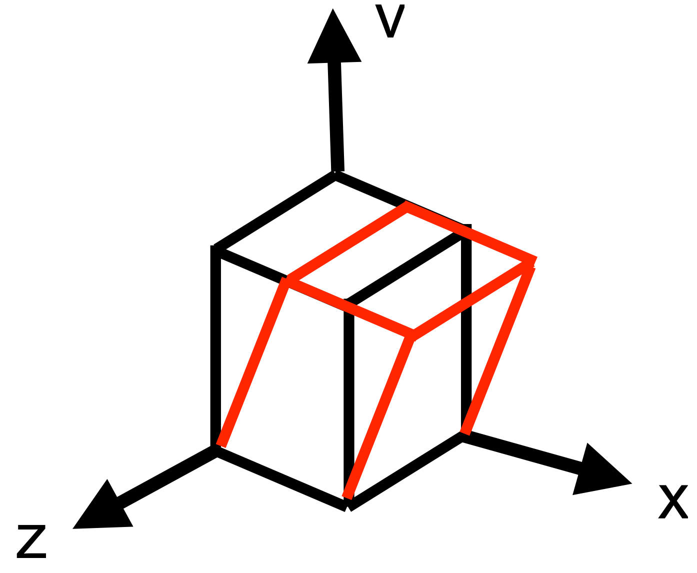
Rotation Expressed in Homogeneous Coordinates

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ \cos\alpha \cdot p_2 - \sin\alpha \cdot p_3 \\ \sin\alpha \cdot p_2 + \cos\alpha \cdot p_3 \end{pmatrix}$$

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix} = \begin{pmatrix} p_1 \\ \cos\alpha \cdot p_2 - \sin\alpha \cdot p_3 \\ \sin\alpha \cdot p_2 + \cos\alpha \cdot p_3 \\ 1 \end{pmatrix}$$

Shearing Expressed in Homogeneous Coordinates

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 + m \cdot p_2 \\ p_2 \\ p_3 \end{pmatrix}$$



$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix} = \begin{pmatrix} p_1 + m \cdot p_2 \\ p_2 \\ p_3 \\ 1 \end{pmatrix}$$

Shearing: General Case

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & m_{12} & m_{13} & 0 \\ m_{21} & 1 & m_{23} & 0 \\ m_{31} & m_{32} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} p_1 + m_{12} \cdot p_2 + m_{13} \cdot p_2 \\ p_2 + m_{21} \cdot p_1 + m_{23} \cdot p_3 \\ p_3 + m_{31} \cdot p_1 + m_{32} \cdot p_2 \\ 1 \end{pmatrix}$$

Computational Complexity for 3D Transformations

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \cdot p_1 + a_{12} \cdot p_2 + a_{13} \cdot p_3 + t_1 \\ a_{21} \cdot p_1 + a_{22} \cdot p_2 + a_{23} \cdot p_3 + t_2 \\ a_{31} \cdot p_1 + a_{32} \cdot p_2 + a_{33} \cdot p_3 + t_3 \\ 1 \end{pmatrix}$$

- Operations needed:
  - 9 multiplications
  - 9 additions
- … for an arbitrarily complex affine 3D transformation
- Runtime complexity improved by pre-calculation of composed transformation matrices
  - Hardware implementations in graphics processors

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Combining several transformations: Order matters!

$$p' = A \times B \times p = A \times (B \times p) = (A \times B) \times p \neq (B \times A) \times p$$

$$p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A = Rotation 90° around X axis (i.e. Y becomes Z ;-)
B = Translation by 5 along Y axis

ABp = A(Bp) means: first translate, then rotate the result
BAp = B(Ap) means: first rotate, then translate the result

$$(A \times B) \times p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 1 \end{pmatrix}$$

$$(B \times A) \times p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \\ 1 \end{pmatrix}$$

The same example in Three.js

```
var p = new THREE.Vector4( 1, 0, 0, 1);

var m = new THREE.Matrix4(); // initialized by identity

var a = new THREE.Matrix4();
var b = new THREE.Matrix4();

var gamma = Math.PI/2; // equals 90 degrees

a.makeRotationX( gamma ); // rotation by 90 degrees around X axis
b.makeTranslation( 0, 5, 0 ); // translation by 5 along Y axis

m.multiply( a ); // Now m contains a
m.multiply( b ); // Now m contains ab

p.applyMatrix4( m ); // Now p contains abp
```